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Exact second moment of oscillator displacement with randomly modulated frequency

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Abstract. A stochastic differential equation with random frequency and driving force is investigated. The exact first and second moment of the solution of the equation are obtained by means of an elementary mathematical method, namely, iteration solving the differential equation in a short time. Some results are consistent with earlier known studies, such as an equilibrium value of the second moment for constant frequency and the divergence of the value for random frequency, by considering limiting cases. It is also found that the transient behaviour of the moments greatly depends on parameters especially when the equilibrium second moment is near the divergence point.

1. Introduction

The problem discussed in this paper is stochastic behaviour of an oscillator displacement $x(t)$ which is subject to the following linear stochastic differential equation:

$$d^2x/dt^2 + \alpha\, dx/dt + \omega^2(t)x = f(t) \tag{1}$$

where α is a deterministic constant parameter and $\omega(t)$ and $f(t)$ are stochastic variables corresponding to a fluctuating frequency and driving force, respectively. To accomplish an exact statistical treatment of the equation, we assume that $\omega(t)$ and $f(t)$ cannot vary with time except when t is at an integral number multiples of Δt_f or Δt_ω and that there are neither time correlation nor correlation between $\omega(t)$ and $f(t)$. $\omega(t)$ and $f(t)$ can then be written as

$$f(t) = f_l \qquad \text{for} \quad \Delta t_f(l-1) \leqslant t < \Delta t_f l \tag{2}$$

$$\omega(t) = \omega_l \qquad \text{for} \quad \Delta t_\omega(l-1) \leqslant t < \Delta t_\omega l \tag{3}$$

where l is an integer. If we write

$$\omega_l^2 = \omega_0^2(1+\epsilon_l) \tag{4}$$

and take

$$\langle f(t)\rangle = 0 \tag{5}$$

$$\langle \epsilon_l \rangle = 0 \tag{6}$$

then

$$\langle f_i f_j \rangle = f^2 \delta_{ij}/\Delta t_f \tag{7}$$

$$\langle \omega_i^2 \omega_j^2 \rangle = \omega_0^4(1 + \langle \epsilon^2 \rangle \delta_{ij}) \tag{8}$$

where the bracketed terms denote ensemble averages and i and j are integers. Δt_f is required as the denominator in (7) to maintain the condition $\sum_i \langle f_i f_j \rangle \Delta t_f = f^2$, which is needed to obtain the limiting expression calculated in sections 3 and 4. A more detailed description of $\omega(t)$ and $f(t)$ will be provided in the following sections.

Over the last few decades, attempts to find a stochastic property of $\omega(t)$ and $f(t)$ such that an exact equilibrium value of the moment $\langle x^2 \rangle_{eq}$ is obtained have been successful. For instance, Bourret [1] assumed a two-valued Markov process for the frequency $\omega(t)$ and West [2] assumed delta-correlated cumulants [3] of the frequency $\omega(t)$. It was found in those papers that the effects of the frequency fluctuation results in renormalization of the parameter α and in energetic instability [4], i.e. $\langle x^2 \rangle_{eq}$ can be infinity when some parameters become particular values. To find such exact results, however, a very refined mathematical approach must be required, therefore, it is very difficult to find another example as successful. In this paper, the need for advanced mathematical equipment is avoided by the abandonment of considering continuous changing of $\omega(t)$ and $f(t)$. It may seem that the method described in this paper is elementary; however, it is possible to develop a method for more complicated cases, step by step, without finding an advanced mathematical idea. Furthermore, the method has other advantages such as making it possible to obtain not only equilibrium values of the second moment $\langle x^2 \rangle_{eq}$ but also transient values of the second moment $\langle x^2(t) \rangle$. Obtaining such transient values of the second moment might be important for awareness of the dynamics of the system.

It has long been suggested that stochastic equations can be used for investigating several kinds of physical and technological systems such as wave propagation in continuous random media, scattering of waves by randomly distributed scatters, control theory and so on [5–8]. Especially in recent years, the study of stochastic differential equations involving both multiplicative and additive noise has caught attention, such as the dye laser model, polymers in turbulent flow and so on [9–11]. Power-law behaviour of the probability distribution is sometimes inferred, which should be related to the divergence of the moments mentioned above [12–14]. Such behaviour is widely seen in actual phenomena. For example, in cases such as the frequency of jams in internet traffic [15], polymer conformations [11], stock market price changes [16] and so on, therefore a general understanding of the power law is essential. In this paper, although the existence of power-law behaviour of the probability distribution function is not proved, divergence behaviour of the second moment and a condition for it is derived. Two different noises were also introduced by another physical reasoning such that internal noise and external noise of the system were to be considered separately [17, 18]. Hopefully this paper will become a starting point for the understanding of this broad area of research as just previously mentioned.

2. Our method of investigation

In this section, fundamental equations to obtain the moments $\langle x(t) \rangle$ and $\langle x(t)^2 \rangle$ are derived. As a beginning, let us consider the case of constant ω, and after that, extend the method to the case of variable ω.

From the assumption of a constant driving force (2) in the time range $0 \leqslant t < \Delta t_f$, $x(\Delta t_f)$ is obtained as a general solution of a second-order ordinary differential equation with constant coefficients as follows.

For $\alpha^2 - 4\omega^2 > 0$,

$$x(\Delta t_f) = \mathrm{e}^{-\alpha \Delta t_f/2}(a \exp(-(\alpha^2 - 4\omega^2)^{1/2} \Delta t_f/2) + b \exp((\alpha^2 - 4\omega^2)^{1/2} \Delta t_f/2)) + f_1/\omega^2 \tag{9}$$

and for $\alpha^2 - 4\omega^2 < 0$,

$$x(\Delta t_f) = \mathrm{e}^{-\alpha\Delta t_f/2}(a\exp(\mathrm{i}(4\omega^2-\alpha^2)^{1/2}\Delta t_f/2) + b\exp(-\mathrm{i}(4\omega^2-\alpha^2)^{1/2}\Delta t_f/2)) + f_1/\omega^2 \tag{10}$$

where i is pure imaginary and where a and b are constants determined by initial condition $x(0)$ and $\mathrm{d}x(0)/\mathrm{d}t$, that is, for the case of $\alpha^2 - 4\omega^2 > 0$,

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} (-\alpha+k)/2k & -1/k \\ (\alpha+k)/2k & 1/k \end{pmatrix} \begin{pmatrix} x(0) \\ \mathrm{d}x(0)/\mathrm{d}t \end{pmatrix} - f_1/\omega^2 \begin{pmatrix} (-\alpha+k)/2k \\ (\alpha+k)/2k \end{pmatrix} \tag{11}$$

$$k = \sqrt{\alpha^2 - 4\omega^2} \tag{12}$$

and for the case of $\alpha^2 - 4\omega^2 < 0$,

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} (\alpha+\mathrm{i}k)/2\mathrm{i}k & 1/2\mathrm{i}k \\ (-\alpha+\mathrm{i}k)/2\mathrm{i}k & -1/2\mathrm{i}k \end{pmatrix} \begin{pmatrix} x(0) \\ \mathrm{d}x(0)/\mathrm{d}t \end{pmatrix} - f_1/\omega^2 \begin{pmatrix} (\alpha+\mathrm{i}k)/2\mathrm{i}k \\ (-\alpha+\mathrm{i}k)/2\mathrm{i}k \end{pmatrix} \tag{13}$$

$$k = \sqrt{4\omega^2 - \alpha^2}. \tag{14}$$

From (9), (10) and their differentials,,

$$\boldsymbol{x}(\Delta t_f) = \begin{pmatrix} x(\Delta t_f) \\ \mathrm{d}x(\Delta t_f)/\mathrm{d}t \end{pmatrix}$$

can be written as linear transformation of $\begin{pmatrix} a \\ b \end{pmatrix}$, so that $\boldsymbol{x}(\Delta t_f)$ can also be written as a linear transformation of

$$\boldsymbol{x}(0) = \begin{pmatrix} x(0) \\ \mathrm{d}x(0)/\mathrm{d}t \end{pmatrix}$$

by using (11) or (13). Therefore, a relation between $\boldsymbol{x}(\Delta t_f)$ and $\boldsymbol{x}(0)$ is written in the following form:

$$\boldsymbol{x}(\Delta t_f) = M\boldsymbol{x}(0) + f_1\boldsymbol{v}. \tag{15}$$

If we define $\boldsymbol{x}_l$ as

$$\boldsymbol{x}_l = \boldsymbol{x}(t = \Delta t_f l) = \begin{pmatrix} x(t = \Delta t_f l) \\ \mathrm{d}x(t = \Delta t_f l)/\mathrm{d}t \end{pmatrix} \tag{16}$$

where l is an integer, then the values of the components of $\boldsymbol{x}_l$ corresponding to a sample sequence of the noise $\{f_1, f_2, \ldots, f_l\}$ can be obtained by iteration of the same transformation to (15) since the relation between $\boldsymbol{x}_{l+1}$ and $\boldsymbol{x}_l$ is the same as the relation between $\boldsymbol{x}_1$ and $\boldsymbol{x}_0$ for all l.

After diagonalization, we obtain

$$\boldsymbol{y}_l = S\boldsymbol{y}_{l-1} + f_l\boldsymbol{u} \tag{17}$$

where S, U, $\boldsymbol{u}$ and $\boldsymbol{y}_l$ are defined and calculated as follows:

$$S = \begin{pmatrix} \lambda^{(+)} & 0 \\ 0 & \lambda^{(-)} \end{pmatrix} = U^{-1}MU \tag{18}$$

$$\lambda^{(\pm)} = \exp\left((-\alpha \pm k)\Delta t_f/2\right) \tag{19}$$

$$U = \begin{pmatrix} 1 & 1 \\ (k-\alpha)/2 & (-k-\alpha)/2 \end{pmatrix} \tag{20}$$

for $\alpha^2 - 4\omega^2 > 0$ and

$$\lambda^{(\pm)} = \exp\left((-\alpha \pm \mathrm{i}k)\Delta t_f/2\right) \tag{21}$$

$$U = \begin{pmatrix} 1 & 1 \\ (\mathrm{i}k - \alpha)/2 & (-\mathrm{i}k - \alpha)/2 \end{pmatrix} \tag{22}$$

for $\alpha^2 - 4\omega^2 < 0$.

$$\boldsymbol{y}_l \equiv \begin{pmatrix} y_l^{(1)} \\ y_l^{(2)} \end{pmatrix} = U^{-1}\boldsymbol{x}_l \tag{23}$$

$$\boldsymbol{u} = \begin{pmatrix} u^{(1)} \\ u^{(2)} \end{pmatrix} = U^{-1}\boldsymbol{v}$$

$$= 1/2k\omega^2 \begin{pmatrix} (-k-\alpha)(\exp((k-\alpha)\Delta t_f/2) - 1) \\ (-k+\alpha)(\exp((-k-\alpha)\Delta t_f/2) - 1) \end{pmatrix} \qquad \text{for} \quad \alpha^2 > 4\omega^2 \tag{24}$$

$$= \mathrm{i}/4k\omega^2 \begin{pmatrix} (\alpha+\mathrm{i}k)(\exp((\mathrm{i}k-\alpha)\Delta t_f/2) - 1) \\ (-\alpha+\mathrm{i}k)(\exp((-\mathrm{i}k-\alpha)\Delta t_f/2) - 1) \end{pmatrix} \qquad \text{for} \quad 4\omega^2 > \alpha^2. \tag{25}$$

By solving (17), the expression of $y_l^{(1)}$ and $y_l^{(2)}$ can be written in the following summation form:

$$y_l^{(1)} = (\lambda^{(+)})^l y_0^{(1)} + u^{(1)} \sum_{j=1}^{l} (\lambda^{(+)})^{l-j} f_j \tag{26}$$

$$y_l^{(2)} = (\lambda^{(-)})^l y_0^{(2)} + u^{(2)} \sum_{j=1}^{l} (\lambda^{(-)})^{l-j} f_j. \tag{27}$$

Equations (26) and (27) are the first fundamental equations presented in this paper. These shall be used in the derivation of the moments for the case of constant ω in section 3.

The expressions of x_l and $(\mathrm{d}x/\mathrm{d}t)_l$ are obtained by the inverse transformation of (23). The moments $\langle x_l^2\rangle$ and $\langle x_l\rangle$ are also obtained through averaging an calculation, which will be written in terms of various kinds of moments of f_l. The statistical treatments of the results will be put off until the next section and the extension of the method to the case of variable ω will be briefly carried out.

Let $\omega(t)$ be a random variable whose value can change only when t is an integer times a time interval Δt_ω. For simple computation, let Δt_ω be an integer times Δt_f, i.e.

$$\Delta t_\omega = m\Delta t_f \tag{28}$$

where m is a constant integer. This assumption applies for a situation where the frequency fluctuation is slower than the changing rate of the driving force. The constant integer m must be physically selected so as to express the changing rate of $\omega(t)$. In fact, if Δt_f is very small Δt_ω can be adjusted almost freely.

In a limited time range such as $0 \leqslant t < \Delta t_\omega$, the situation does not alter from the case of constant ω. Therefore, if we introduce $Y_1^{(1)} = y_{1\times m}^{(1)}$ and $Y_1^{(2)} = y_{1\times m}^{(2)}$, the recursion relations of $Y_n^{(1)}$ and $Y_n^{(2)}$ would seem to be obtained from (26) and (27). However, it should be noted that the transformation of (23) must be performed according to a randomly changed matrix

U in each time interval. If we introduce the transformed variable $\boldsymbol{Y}_n = U_{n+1}^{-1}\boldsymbol{x}_{n\times m}$, this next formula holds from (26), (27) and the inverse transformation of (23),

$$\begin{aligned}\boldsymbol{Y}_n &= U_{n+1}^{-1}\boldsymbol{X}_n \\ &= U_{n+1}^{-1}\{U_n\Lambda_n\boldsymbol{Y}_{n-1} + U_n\boldsymbol{F}_n\}\end{aligned} \tag{29}$$

where we have defined $\boldsymbol{X}_n$ and other variables such as

$$\boldsymbol{X}_n = \begin{pmatrix} X_n \\ \dot{X}_n \end{pmatrix} = \begin{pmatrix} x_{n\times m} \\ \dot{x}_{n\times m} \end{pmatrix} \tag{30}$$

$$\Lambda_n = \begin{pmatrix} (\lambda_n^{(+)})^m & 0 \\ 0 & (\lambda_n^{(-)})^m \end{pmatrix} = \begin{pmatrix} \Lambda_n^{(+)} & 0 \\ 0 & \Lambda_n^{(-)} \end{pmatrix} \tag{31}$$

and

$$\boldsymbol{F}_n = \begin{pmatrix} F_n^{(1)} \\ F_n^{(2)} \end{pmatrix} = \begin{pmatrix} u_n^{(1)}\sum_{j=1}^m (\lambda_n^{(+)})^{m-j} f_{(n\times m)+j} \\ u_n^{(2)}\sum_{j=1}^m (\lambda_n^{(-)})^{m-j} f_{(n\times m)+j} \end{pmatrix}. \tag{32}$$

The reason why the subscript n is used in these definitions is because these values are dependent on frequency fluctuation ω_n. Statistical calculation based on this formulae will be performed in section 4.

The non-diagonal matrix in (29)

$$U_{n+1}^{-1}U_n = \begin{pmatrix} (k_{n+1}+k_n)/2k_{n+1} & (k_{n+1}-k_n)/2k_{n+1} \\ (k_{n+1}-k_n)/2k_{n+1} & (k_{n+1}+k_n)/2k_{n+1} \end{pmatrix} \tag{33}$$

represents the difference between the case of random and constant ω and how the two decreasing modes $Y^{(1)}$ and $Y^{(2)}$ are related. However, as a very rough approximation, we can consider the matrix (33) as the identity matrix when the frequency fluctuation intensity is small. Under such an approximation, the recursion relation about $Y_n^{(1)}$ and $Y_n^{(2)}$ can be derived as before because of the diagonalized form of equation (29). According to the approximation, $Y_n^{(1)}$ and $Y_l^{(2)}$ can be written as

$$Y_n^{(1)} = \prod_{t=1}^{n}(\Lambda_t^{(+)})Y_0^{(1)} + \sum_{j=1}^{n-1}\prod_{t=0}^{j-1}(\Lambda_{n-t}^{(+)})F_{n-j}^{(1)} + F_n^{(1)} \tag{34}$$

$$Y_l^{(2)} = \prod_{t=1}^{n}(\Lambda_t^{(-)})Y_0^{(2)} + \sum_{j=1}^{n-1}\prod_{t=0}^{j-1}(\Lambda_{n-t}^{(-)})F_{n-j}^{(2)} + F_n^{(2)}. \tag{35}$$

The expressions (34) and (35) are not exact but are sometimes meaningful to find qualitative results easily, which will be briefly described in section 4.

All results derived in this section may be useful when considering various kinds of random variables f_l and ω_n^2 with correlation. However, in the following sections in which the calculation of moments $\langle X_n^2\rangle$ and $\langle X_n\rangle$ will be performed, f's and ω's are assumed to be independent random variables as described in section 1 in order to find exact results easily.

3. Results for constant ω

In this section, moments of the displacement x_l are calculated and examined for the case of constant ω.

Equation (17) yields the following formulae:

$$y_l^{(1)} = \lambda^{(+)} y_{l-1}^{(1)} + f_l u^{(1)} \tag{36}$$

$$(y_l^{(1)})^2 = (\lambda^{(+)} y_{l-1}^{(1)})^2 + 2\lambda^{(+)} y_{l-1}^{(1)} f_l u^{(1)} + (f_l u^{(1)})^2. \tag{37}$$

Using (5) and the statistical independence between f_l and $y_{l-1}^{(1)}$ we obtain through averaging of (36) and (37),

$$\langle y_l^{(1)} \rangle = \lambda^{(+)} \langle y_{l-1}^{(1)} \rangle \tag{38}$$

$$\langle (y_l^{(1)})^2 \rangle = (\lambda^{(+)})^2 \langle (y_{l-1}^{(1)})^2 \rangle + (u^{(1)})^2 \langle (f_l)^2 \rangle. \tag{39}$$

The equilibrium value $\lim_{l\to\infty}\langle x_l \rangle = \langle x \rangle_{eq}$ and $\lim_{l\to\infty}\langle x_l^2 \rangle = \langle x^2 \rangle_{eq}$ can be handled easily. Considering the property such that $\langle y^{(1)} \rangle_{eq} = \langle y_{l+1}^{(1)} \rangle = \langle y_l^{(1)} \rangle$ for large l, equation (38) yields $\langle y^{(1)} \rangle_{eq} = 0$. Since a similar calculation yields $\langle y^{(2)} \rangle_{eq} = 0$, $\langle x \rangle_{eq}$ is obtained as 0 by using (23), which is a trivial result. Considering the property such that $\langle (y^{(1)})^2 \rangle_{eq} = \langle (y_{l+1}^{(1)})^2 \rangle = \langle (y_l^{(1)})^2 \rangle$ for large l, equation (39) yields

$$\langle (y^{(1)})^2 \rangle_{eq} = (u^{(1)})^2 \langle (f_l)^2 \rangle / (1 - (\lambda^{(+)})^2). \tag{40}$$

One can obtain through a similar calculation,

$$\langle y^{(1)} y^{(2)} \rangle_{eq} = u^{(1)} u^{(2)} \langle (f_l)^2 \rangle / (1 - \lambda^{(+)} \lambda^{(-)}) \tag{41}$$

$$\langle (y^{(2)})^2 \rangle_{eq} = (u^{(2)})^2 \langle (f_l)^2 \rangle / (1 - (\lambda^{(-)})^2) \tag{42}$$

where explicit forms of $\langle (f_l)^2 \rangle$, $u^{(1)}$, $u^{(2)}$, $\lambda^{(+)}$ and $\lambda^{(-)}$ are given in (7), (19), (21), (24) and (25). For example, when $\alpha^2 > 4\omega^2$, $(\lambda^{(\pm)})^2$ and $(\lambda^{(+)}\lambda^{(-)})$ can be rewritten by (19) as

$$(\lambda^{(\pm)})^2 = \exp((-\alpha \pm k)\Delta t_f) \tag{43}$$

$$(\lambda^{(+)}\lambda^{(-)}) = \exp(-\alpha \Delta t_f). \tag{44}$$

The equilibrium value of the second moment $\langle x^2 \rangle_{eq}$ can be obtained by using (23) and substituting these explicit forms into (40)–(42) as

$$\begin{aligned}\langle x^2 \rangle_{eq} &= \langle (y^{(1)})^2 \rangle_{eq} + 2\langle y^{(1)} y^{(2)} \rangle_{eq} + \langle (y^{(2)})^2 \rangle_{eq} \\ &= (f^2/4k^2\omega^4 \Delta t_f)\big\{(k+\alpha)^2(\exp((k-\alpha)\Delta t_f/2) - 1)^2/(1 - \exp((k-\alpha)\Delta t_f)) \\ &\quad -8\omega^2(\exp((k-\alpha)\Delta t_f/2) - 1) \\ &\quad \times(\exp((-k-\alpha)\Delta t_f/2) - 1)/(1 - \exp(-\alpha\Delta t_f)) \\ &\quad +(\alpha - k)^2(\exp((-k-\alpha)\Delta t_f/2) - 1)^2/(1 - \exp((-k-\alpha)\Delta t_f))\big\}\end{aligned} \tag{45}$$

for $\alpha^2 - 4\omega^2 > 0$ and

$$\begin{aligned}\langle x^2 \rangle_{eq} &= (f^2/2\omega^2 k^2 \Delta t_f)\{A/(1 + \exp(-\alpha\Delta t_f) + 2\exp(-\alpha\Delta t_f/2)\cos(k\Delta t_f/2)) \\ &\quad +B/(1 - \exp(-\alpha\Delta t_f))\}\end{aligned} \tag{46}$$

for $\alpha^2 - 4\omega^2 < 0$, where

$$A = (k^2 - \alpha^2)(1 - \exp(-\alpha\Delta t_f)) - 4k\alpha\exp(-\alpha\Delta t_f/2)\sin(k\Delta t_f/2) \tag{47}$$

$$B = -4\omega^2(-\exp(-\alpha\Delta t_f) + 2\exp(-\alpha\Delta t_f/2)\cos(k\Delta t_f/2) - 1). \tag{48}$$

Noting that expressions (40) and (42) are complex conjugates makes the derivation of (46) easier. This situation is similar to the derivation of (60).

To examine these results, let us consider the limiting case of small Δt_f such that

$$\left(e^{-A\Delta t_f} - 1\right)\left(e^{-B\Delta t_f} - 1\right) / \left(e^{-C\Delta t_f} - 1\right) \simeq AB\Delta t_f / C + \left\{AB - \left(BA^2 + AB^2\right)/C\right\} \Delta t_f^2/2. \quad (49)$$

By applying this limiting procedure to (45) or (46), we obtain

$$\lim_{\Delta t_f \to 0} \langle x^2 \rangle_{eq} = f^2/2\alpha\omega^2 + \mathrm{o}\left((\Delta t_f)^2\right) \quad (50)$$

which holds in either cases of $\alpha^2 - 4\omega^2 > 0$ or $\alpha^2 - 4\omega^2 < 0$. Formula (50) agrees with a well known result obtained by Uhlenbeck and Ornstein [19, 20], which is an expected result since the right-hand side of (7) becomes a delta function as Δt_f tends to 0. Since the equation is linear and the noise is additive, its distribution should be entirely characterized by these first and second moments. However, this property will be changed by frequency fluctuation. The linear term in (50) with respect to Δt_f is zero, which indicates that the increase of $\langle x^2 \rangle_{eq}$ is hardly caused by a small Δt_f.

In the limiting case of $\Delta t_f \to \infty$, the right-hand side of (45) and (46) becomes the average value of equilibrium x^2 determined by the deterministic differential equation, that is $\langle f^2 \rangle / \omega^4$, which tends to 0 as $\Delta t_f \to \infty$ under the assumption of (7).

The time dependence of $\langle x_l \rangle$ and $\langle x_l^2 \rangle$ can also be derived from (38) and (39) as follows. With (38) and (39), $\langle y_l^{(1)} \rangle$ and $\langle (y_l^{(1)})^2 \rangle - \langle (y^{(1)})^2 \rangle_{eq}$ are geometric series with an equal ratio of $(\lambda^{(+)})$ and $(\lambda^{(+)})^2$, respectively. Hence,

$$\langle y_l^{(1)} \rangle = (\lambda^{(+)})^l \langle y_0^{(1)} \rangle \quad (51)$$

$$\langle (y_l^{(1)})^2 \rangle = (\lambda^{(+)})^{2l} (\langle (y_0^{(1)})^2 \rangle - \langle (y^{(1)})^2 \rangle_{eq}) + \langle (y^{(1)})^2 \rangle_{eq}). \quad (52)$$

A similar computation can be carried out for $\langle y_l^{(2)} \rangle$, $\langle y_l^{(1)} y_l^{(2)} \rangle$ and $\langle (y_l^{(2)})^2 \rangle$. Then, $\langle x_l \rangle$ and $\langle x_l^2 \rangle$ can be obtained by using (23) as

$$\langle x_l \rangle = (\lambda^{(+)})^l \langle y_0^{(1)} \rangle + (\lambda^{(-)})^l \langle y_0^{(2)} \rangle \quad (53)$$

$$\begin{aligned} \langle x_l^2 \rangle = {} & (\lambda^{(+)})^{2l} (\langle (y_0^{(1)})^2 \rangle - \langle (y^{(1)})^2 \rangle_{eq}) + 2(\lambda^{(+)}\lambda^{(-)})^l (\langle y_0^{(1)} y_0^{(2)} \rangle - \langle y^{(1)} y^{(2)} \rangle_{eq}) \\ & + (\lambda^{(-)})^{2l} (\langle (y_0^{(1)})^2 \rangle - \langle (y^{(1)})^2 \rangle_{eq}) + \langle x^2 \rangle_{eq}. \end{aligned} \quad (54)$$

Formula (54) depends on the initial conditions of $y_0^{(1)}$ and $y_0^{(2)}$, which are determined by initial conditions of $\boldsymbol{x}_0$. For example, for the case of $\alpha^2 - 4\omega^2 > 0$, from (23)

$$(y_0^{(1)})^2 = ((-k - \alpha)x_0/2 - (\mathrm{d}x/\mathrm{d}t)_0)^2/k^2 \quad (55)$$

$$y_0^{(1)} y_0^{(2)} = ((-k - \alpha)/2 - (\mathrm{d}x/\mathrm{d}t)_0)((-k + \alpha)/2 + (\mathrm{d}x/\mathrm{d}t)_0)/k^2 \quad (56)$$

$$(y_0^{(2)})^2 = ((-k + \alpha)/2 + (\mathrm{d}x/\mathrm{d}t)_0)^2/k^2. \quad (57)$$

Finally, considering the time dependence of variance $\langle (x_l - \langle x_l \rangle)^2 \rangle$. It can be easily seen that $\langle (x_l - \langle x_l \rangle)^2 \rangle$ is independent of the initial condition $\boldsymbol{x}_0$. From (23), (26), (27) and (53), if the initial condition have no distribution,

$$\begin{aligned} x_l - \langle x_l \rangle &= y_l^{(1)} + y_l^{(2)} - \langle y_l^{(1)} + y_l^{(2)} \rangle \\ &= u^{(1)} \sum_{j=1}^{l} (\lambda^{(+)})^{l-j} f_j + u^{(2)} \sum_{j=1}^{l} (\lambda^{(-)})^{l-j} f_j. \end{aligned} \quad (58)$$

This equation is independent of initial conditions. Averaging the square of (58) yields,

$$\begin{aligned}\langle (x_l - \langle x_l \rangle)^2 \rangle &= \langle f_l^2 \rangle \{(u^{(1)})^2 (1 - (\lambda^{(+)})^{2l})/(1 - (\lambda^{(+)})^2) \\ &\quad + 2u^{(1)} u^{(2)} (1 - (\lambda^{(+)})^l (\lambda^{(-)})^l)/(1 - \lambda^{(+)} \lambda^{(-)}) \\ &\quad + (u^{(2)})^2 (1 - (\lambda^{(-)})^{2l})/(1 - (\lambda^{(-)})^2)\} \\ &= \langle (y^{(1)})^2 \rangle_{eq} (1 - (\lambda^{(+)})^{2l}) + 2 \langle y^{(1)} y^{(2)} \rangle_{eq} (1 - (\lambda^{(+)})^l (\lambda^{(-)})^l) \\ &\quad + \langle y^{(2)} \rangle_{eq} (1 - (\lambda^{(-)})^{2l}) \end{aligned} \tag{59}$$

where cross terms have been vanished from (7). Instead (40)–(42) have been used. It is observed from (59), (43) and (44) that, for the case of $\alpha^2 - 4\omega^2 > 0$, there are three modes of approaching the equilibrium value of $\langle x^2 \rangle_{eq}$ such as $\exp(-\alpha \pm k)\Delta t_f l$ and $\exp(-\alpha)\Delta t_f l$. While, for the case of $\alpha^2 - 4\omega^2 < 0$, substitution of (21) for $\lambda^{(\pm)}$ of (59) yields (refer to below equation (48))

$$\begin{aligned}\langle (x_i - \langle x_l \rangle)^2 \rangle &= 2\{\mathrm{Im}\{\langle (y^{(1)})^2 \rangle_{eq}\} \sin k\Delta t_f l - \mathrm{Re}\{\langle (y^{(1)})^2 \rangle_{eq}\} \cos k\Delta t_f l\} \exp(-\alpha \Delta t_f l) \\ &\quad - 2\langle y^{(1)} y^{(2)} \rangle_{eq} \exp(-\alpha \Delta t_f l) + \langle x^2 \rangle_{eq}. \end{aligned} \tag{60}$$

It is interesting that not only can the mean value of x_i oscillate, but so does the order of displacement from its mean value.

4. Results for a variable ω

In this section, the displacement moments $\langle x^2(t) \rangle$ are calculated and examined for the case of variable ω. The most fundamental equation for this case is (29), from which we can derive the following equation by multiplying U_{n+1} on the left:

$$\boldsymbol{X}_n = L_n \boldsymbol{X}_{n-1} + \tilde{\boldsymbol{F}}_n \tag{61}$$

where the matrix L_n and the vector $\tilde{\boldsymbol{F}}_n$ have been defined as

$$\begin{aligned} L_n &= U_n \Lambda_n U_n^{-1} \\ &= \begin{pmatrix} (\alpha + k_n)\Lambda_n^{(+)}/2k_n + (\alpha - k_n)\Lambda_n^{(+)}/2k_n & \Lambda_n^{(+)}/k_n - \Lambda_n^{(-)}/k_n \\ -\omega_n^2 \Lambda_n^{(+)}/k_n + \omega_n^2 \Lambda_n^{(-)}/k_n & (\alpha - k_n)\Lambda_n^{(+)}/2k_n + (\alpha + k_n)\Lambda_n^{(+)}/2k_n \end{pmatrix} \end{aligned} \tag{62}$$

$$\tilde{\boldsymbol{F}}_n \equiv U_n \boldsymbol{F}_n. \tag{63}$$

Noting statistical independence between $\boldsymbol{X}_n$ and L_n and between $\boldsymbol{F}_n$ and of L_n and the property that $\langle \boldsymbol{F}_n \rangle$ equals zero, which is derived from (5) and (32), we can derive recursion relation for the moments after an averaging procedure for each squared element of $\boldsymbol{X}_n$ in (61),

$$\begin{aligned} \begin{pmatrix} \langle X_n^2 \rangle \\ \langle X_n \dot{X}_n \rangle \\ \langle \dot{X}_n^2 \rangle \end{pmatrix} &= \begin{pmatrix} \langle (L_n^{(11)})^2 \rangle & 2\langle L_n^{(11)} L_n^{(12)} \rangle & \langle (L_n^{(12)})^2 \rangle \\ \langle L_n^{(11)} L_n^{(21)} \rangle & \langle L_n^{(12)} L_n^{(21)} + L_n^{(11)} L_n^{(22)} \rangle & \langle L_n^{(12)} L_n^{(22)} \rangle \\ \langle (L_n^{(21)})^2 \rangle & 2\langle L_n^{(21)} L_n^{(22)} \rangle & \langle (L_n^{(22)})^2 \rangle \end{pmatrix} \begin{pmatrix} \langle X_{n-1}^2 \rangle \\ \langle X_{n-1} \dot{X}_{n-1} \rangle \\ \langle \dot{X}_{n-1}^2 \rangle \end{pmatrix} \\ &\quad + \begin{pmatrix} \langle (\tilde{F}_n^{(1)})^2 \rangle \\ \langle \tilde{F}_n^{(1)} \tilde{F}_n^{(2)} \rangle \\ \langle (\tilde{F}_n^{(2)})^2 \rangle \end{pmatrix} \end{aligned} \tag{64}$$

$$\equiv \mathcal{L} \vec{\mathcal{X}}_{n-1} + \mathcal{F}. \tag{65}$$

In this equation the subscript n has been omitted in $\mathcal{L}$ and $\mathcal{F}$ because of the stationary property.

We can discuss several exact statistical properties of the displacement moment starting from (65). At first, from the fact that the equilibrium second moment is expected to have the property $\vec{\mathcal{X}}_n = \vec{\mathcal{X}}_{n-1} \equiv \vec{\mathcal{X}}_{eq}$, the formula

$$\mathcal{X}_{eq} = (I - \mathcal{L})^{-1}\mathcal{F} \tag{66}$$

is derived where $(I - \mathcal{L})^{-1}$ means the inverse of the matrix $(I - \mathcal{L})$. This is the general formula for the second moment under the assumption. Some remarkable results such as divergence behaviour can be derived from it in the following. For a simple calculation of $\mathcal{X}_{eq}$, we introduce further assumption that $\Delta t_f \simeq 0$ and ω_n^2 can only take two values

$$\omega_0^2(1 \pm \epsilon) \tag{67}$$

with equal probability. Under such an assumption, the average over frequency fluctuations at one time can be obtained by summing the two realizable values corresponding to ω_n^2 divided by two. From this the elements of $(I - \mathcal{L})$ can be obtained. The elements of $\mathcal{F}$ can also be calculated by averaging with regard to the driving force fluctuation, which is similar to getting (40)–(42). In the situation where α^2 is larger than $4\omega^2(t)$,

$$\begin{aligned}\mathcal{F}^{(1)} &= f^2\langle (k_n + \alpha)(1 - (\Lambda_n^{(+)})^2)/4k_n^2\omega_n^2\rangle - 2f^2\langle (1 - \Lambda_n^{(+)}\Lambda^{(-)})/k_n^2\alpha\rangle \\ &\quad + f^2\langle (-k_n + \alpha)(1 - (\Lambda^{(-)})^2)/4k_n^2\omega_n^2\rangle\end{aligned} \tag{68}$$

$$\mathcal{F}^{(2)} = -f^2\langle (1 - (\Lambda_n^{(+)})^2)/2k_n^2\rangle + f^2\langle (1 - \Lambda_n^{(+)}\Lambda^{(-)})/k_n^2\rangle - f^2\langle (1 - (\Lambda^{(-)})^2)/2k_n^2\rangle \tag{69}$$

$$\begin{aligned}\mathcal{F}^{(3)} &= f^2\langle (\alpha - k_n)(1 - (\Lambda_n^{(+)})^2)/4k_n^2\rangle - 2f^2\langle \omega_n^2(1 - \Lambda_n^{(+)}\Lambda^{(-)})/k_n^2\alpha\rangle \\ &\quad + f^2\langle (k_n + \alpha)(1 - (\Lambda^{(-)})^2)/4k_n^2\rangle\end{aligned} \tag{70}$$

where we have used (7) and (49). For the case of $4\omega^2(t) > \alpha^2$, an alternating point occurs on substituting $\mathrm{i}k_n$ for k_n.

In this way, the explicit form of the equilibrium moment can be obtained which can be mathematically involved but is exact. One of the behaviours of the second displacement moment is plotted in figure 1 with respect to ϵ and Δt_ω. Figure 1 indicates that the second moment becomes infinite when Δt_ω is located on a certain interval and ϵ is beyond any critical values. The divergence of the equilibrium second moment reminds us of another example of an oscillator with fluctuating frequency studied in previous papers mentioned in section 1. It is to be noted that the divergence of the equilibrium second moment is related to the behaviour of a determinant $|I - \mathcal{L}|$, because $|I - \mathcal{L}|$ is considered as the denominator of $(I - \mathcal{L})^{-1}$. In figure 2 two cases of the behaviour of $|I - \mathcal{L}|$ with respect to the parameter Δt_ω are plotted, one of which has negative values in some intervals corresponding to the divergence of the equilibrium second moment (refer to figure 3). The behaviour infers that the oscillating property of $|I - \mathcal{L}|$ with respect to Δt_ω is necessary for the divergence of the second moment to exist which can never be seen in the case of $\alpha^2 > 4\omega^2(t)$.

Transient behaviour of first and second moments of the displacement can be examined, respectively, in (61) and (65) because the behaviour is determined by eigenvalues of $\langle L_n\rangle$ and $\mathcal{L}$. The explicit form of eigenvalues of $\langle L_n\rangle$ can easily obtained as

$$Eig_L = \frac{1}{2}\left\{\langle L^{(22)}\rangle + \langle L^{(11)}\rangle \pm \left((\langle L^{(22)}\rangle - \langle L^{(11)}\rangle)^2 + 4\langle L^{(12)}\rangle\langle L^{(21)}\rangle\right)^{1/2}\right\}. \tag{71}$$

The eigenvalues of $\mathcal{L}$ would be more involved, but the calculation can be carried out. In figure 4, the relaxation time before reaching its equilibrium moment is plotted with respect to

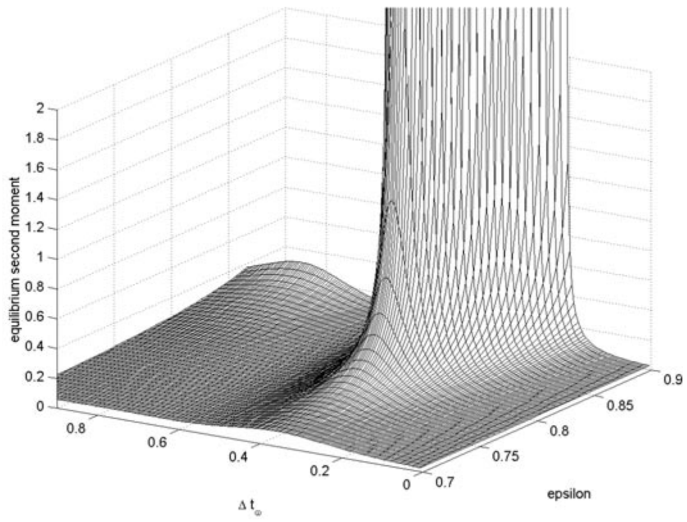

Figure 1. Typical behaviour of the second moment. This plot shows the equilibrium second moment in the limiting case of $\Delta t_f \to 0$ for $\alpha = 1.2$, $\omega_0 = 4$ and $f = 1$. The divergence can take place only when the Δt_ω is in a limited region and ϵ is beyond critical values.

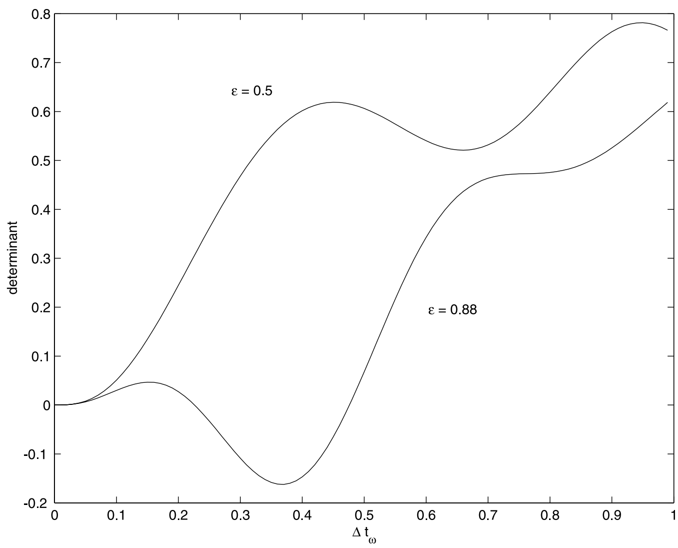

Figure 2. Plot of $|I - \mathcal{L}|$ with respect to the parameter Δt_ω for $\alpha = 1.2$, $\omega_0 = 4$ and $f = 1$. When $\epsilon = 0.88$, $|I - \mathcal{L}|$ is oscillating with a region of negative values, corresponding to the divergence of the equilibrium second moment as shown in figure 3. This figure infers that oscillating property of $|I - \mathcal{L}|$ with respect to Δt_ω is necessary to exist the divergence of the second moment.

ϵ, which has been calculated as $-\Delta t_\omega / \log$ (eigenvalue). It is especially interesting that the sudden increase of the relaxation time is seen in the plot concerning the second moment, which corresponds to divergence of the second moment. The same behaviour is similar to the critical slowing down as seen at phase transitions in thermodynamics. The property that the time order can be far from that of the deterministic original equation by adjusting the parameters Δt_ω or ϵ

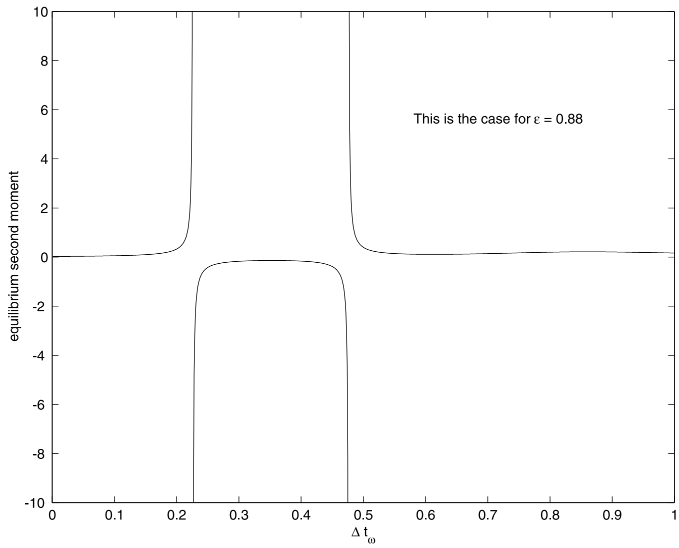

Figure 3. The divergence behaviour of the second moment at $\epsilon = 0.88$ with respect to Δt_ω is indicated here. The second moment becomes infinite corresponding to the negative region of $|I - \mathcal{L}|$ as shown in figure 2. The negative value of the second moment is considered to be physically meaningless.

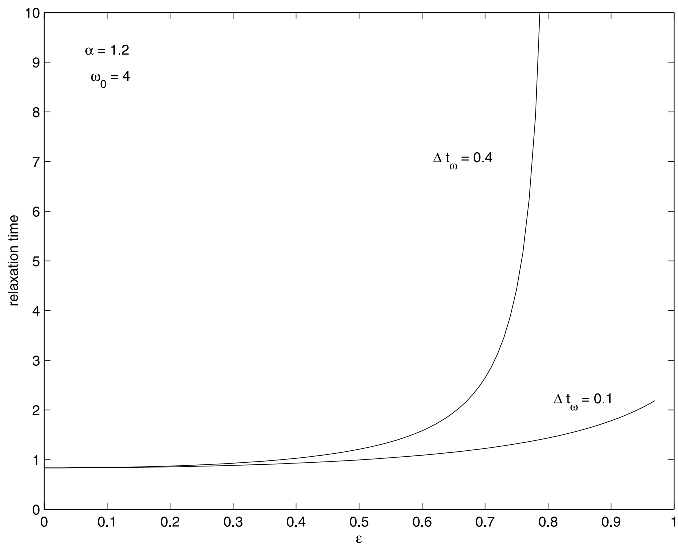

Figure 4. This plot shows the relaxation time calculated from one of the eigenvalues of $\mathcal{L}$ with respect to ϵ when $\alpha = 1.2$, $\omega_0 = 4$. By referring to figure 1, it is found that the relaxation time becomes infinite corresponding to divergence of the second moment. For the case of $\Delta t_\omega = 0.1$, the relaxation time does not become so large, resulting in finite behaviour for the second moment.

describing the property of frequency noise is an essential distinction from the case of constant ω.

Finally, the results derived from the approximate formulae (34) and (35) can be briefly discussed. Some of the approximate results are consistent with exact results described above.

For example, under the condition that $\Delta t_f \simeq 0$ and $\Delta t_\omega = \infty$, both calculations yield the same value of the second moment $\langle 1/2\alpha\omega_n^2\rangle$. Furthermore, under the condition that $\Delta t_f \simeq 0$ and $\Delta t_\omega \simeq 0$, approximate second moments are calculated through a troublesome calculation as for the case of $4\omega^2 < \alpha^2$,

$$\begin{aligned}\lim_{\Delta t_\omega \to 0}\lim_{\Delta t_f \to 0}\langle X^2\rangle_{eq} &= \langle f^2\rangle \Delta t_f (k_+^2 + k_-^2)(k_+ + k_-)^2/\alpha k_+^2 k_-^2 \left\{4\alpha^2 - (k_+ + k_-)^2\right\} \\ &= 2\langle f^2\rangle \Delta t_f (\alpha^2 - 4\omega_0^2) \\ &\quad \times \left\{\alpha^2 - 4\omega_0^2 + 2((\alpha^2 - 4\omega_0^2)^2 - 16\omega_0^4\epsilon^2)^{1/2}\right\}/\alpha\left\{(\alpha^2 - 4\omega_0^2)^2 - 16\omega_0^4\epsilon^2\right\} \\ &\quad \times \left\{\alpha^2 + 4\omega_0^2 - 2((\alpha^2 - 4\omega_0^2)^2 - 16\omega_0^4\epsilon^2)^{1/2}\right\} \end{aligned} \tag{72}$$

and for the case of $4\omega^2 > \alpha^2$,

$$\begin{aligned}\lim_{\Delta t_\omega \to 0}\lim_{\Delta t_f \to 0}\langle X^2\rangle_{eq} &= -\alpha\langle f^2\rangle \Delta t_f \left\{\alpha^2(1/k_+^2\omega_+^2 + 1/k_-^2\omega_-^2)\right. \\ &\quad \left. + (1/\omega_+^2 + 1/\omega_+^2)\right\}/\left\{4\alpha^2 - (k_+ + k_-)^2\right\} + 2\langle f^2\rangle \Delta t_f (4\omega_0^2 - \alpha^2)/\alpha k_+^2 k_-^2 \\ &= 8\langle f^2\rangle \Delta t_f \alpha\left\{4\omega_0^2 + \epsilon^2(\alpha^2 - 4\omega_0^2)\right\}/\left\{4\omega_0^2 + \alpha^2 + ((4\omega_0^2 - \alpha^2)^2 - 16\omega_0^4\epsilon^2)^{1/2}\right\} \\ &\quad \times \left\{(4\omega_0^2 - \alpha^2)^2 - 16\omega_0^4\epsilon^2\right\}(1 - \epsilon^2) \\ &\quad + 2\langle f^2\rangle \Delta t_f (4\omega_0^2 - \alpha^2)/\alpha\left\{(4\omega_0^2 - \alpha^2)^2 - 16\omega_0^4\epsilon^2\right\} \end{aligned} \tag{73}$$

where $k_\pm = \sqrt{\alpha^2 - 4\omega_0^2(1 \pm \epsilon)}$ or $\sqrt{4\omega_0^2(1 \pm \epsilon) - \alpha^2}$. These limiting procedures cannot be interchanged because of the assumption of (28), If $\epsilon = 0$, which means ω does not change with time, this result is, of course, in agreement with equation (50). Furthermore, these moments can be infinite by the limiting procedure $\epsilon \to |\alpha^2/4\omega_0^2 - 1|$. However, this critical point is different from the exact results obtained above.

Transient behaviour of the first and second moment of the displacement can also be discussed starting from (34) and (35). It indicates a difference from the case of constant ω. For example, The expression of $\langle (Y_n^{(1)})^2\rangle$ can be easily obtained in the same way (52) was obtained from (39).

$$\langle (Y_n^{(1)})^2\rangle = \langle (\Lambda^{(+)})^2\rangle^n (\langle (Y_0^{(1)})^2\rangle - \langle (Y^{(1)})^2\rangle_{eq}) + \langle (Y^{(1)})^2\rangle_{eq}. \tag{74}$$

$\langle (\Lambda^{(+)})^2\rangle^n$ can be calculated by substituting (19) for (31), the value of which depends on Δt_ω and ϵ but is not similar to exact results. Another point which can be easily derived from the approximation is the fact that the variance $\langle (X_n - \langle X_n\rangle)^2\rangle$ is dependent on the initial condition of X_0 and $(\mathrm{d}X/\mathrm{d}t)n_0$. This fact is confirmed by the calculation of $Y_n - \langle Y_n\rangle$. From (34) and (35) the expression of $Y_n - \langle Y_n\rangle$ has a term $\prod_{j=1}^n \Lambda_j^{(+)} Y_0^{(1)} - \langle (\Lambda^{(+)})^n\rangle Y_0^{(1)}$, where terms including the expression of $Y_0^{(1)}$ and $Y_0^{(2)}$ do not vanish in the expression of the variance $\langle (X_n - \langle X_n\rangle)^2\rangle$, while for the case of constant ω, where $y_0^{(1)}$ and $y_0^{(2)}$ do not appear at all in (58).

5. Discussion

The moments of the displacement $x(t)$ which is subject to the stochastic differential equation (1) is of principal interest. The main exploitation is to make the analytical integration possible by using noise which varies discretely in time and by assumption of (28). It is a remarkable point that we can extract the effect of the frequency fluctuation in the matrix (33). The results obtained by this idea are enumerated as follows.

In section 3, equilibrium values of the first moment $\langle x \rangle_{eq}$ and the second moment $\langle x^2 \rangle_{eq}$ for the case of constant ω were derived. The result infers that the small time correlation length of the driving force hardly corrects the value of the equilibrium moment (45) which is the well known formula $f^2/(2\alpha\omega^2)$. The transient values of the moment $\langle x_l^2 \rangle$ and the variance $\langle (x_l - \langle x_l \rangle)^2 \rangle$ are also derived as (54), (59) and (60), from which it is found that the relaxation time before reaching to equilibrium value or oscillating state is similar to the original deterministic equation. This behaviour is completely changed by considering frequency fluctuation as discussed in section 4. It is interesting that the transient values of the variance $\langle (x_l - \langle x_l \rangle)^2 \rangle$ can oscillate with time, while it is not surprising that $\langle x_l \rangle$ can oscillate with time just like a solution of the deterministic differential equation.

In section 4, equilibrium values of the second moment $\langle x^2 \rangle_{eq}$ are calculated starting from the exact recursion relation of (29) or approximate relation of (34) and (35). The approximated formula is easy to manipulate, but may sometimes yield quite different results from the correct one. The behaviour of the exact second moment calculated under the assumption of two-valued frequency fluctuations brings about an infinity region of $\langle x^2 \rangle_{eq}$ as shown in figure 1. The divergence behaviour corresponds to earlier known results reviewed in section 1. The most noticeable point we have obtained is that the divergence is related to the value of the determinant $|I - \mathcal{L}|$. This is due to the fact that the divergence point is given by zeros of $|I - \mathcal{L}|$ with respect to Δt_ω as seen in figure 2. However, the explicit form of the divergence point cannot be estimated from $|I - \mathcal{L}|$, which can be mathematically involved. It should be noted that this formulation is correct only when the time correlation of the frequency fluctuation is ignored. It is surprising that the explicit form of $\langle x^2 \rangle_{eq}$ derived from previous works turns into a very simple form which can be interpreted as renormalization of the parameter α. Approximate results also yield the divergence behaviour of second moments as seen in (72) and (73). It indicates that the second moments tend to infinity in the very least case of using the limiting procedure $\lim_{\Delta t_\omega \to 0} \lim_{\Delta t_f \to 0}$. Although both results are in agreement with each other in a qualitative manner, the divergence points clearly occur at different intervals for both the approximate and correct results. The distinction can be better understood from the fact that the approximate results approach is only considered to be valid when the intensity of the frequency fluctuations is relatively small, while an extremely rapid increase of $\langle x^2 \rangle_{eq}$ suddenly takes place when the intense frequency fluctuation becomes large.

Note that the divergence does not always mean that the displacement does not have an equilibrium probability distribution. This is to be understood by the power-law behaviour of the probability distribution as discussed in other papers [12–14]. The power-law behaviour becomes possible, given that the second moment approaches an infinite value. This was also analysed using a Fokker–Planck-type equation, which infers a power-law distribution [21], according to which the distribution does not have the form of a canonical distribution. On the other hand, the considering canonical distribution for the case of only additive noise, the divergence of the second moment may be related to the transition from a canonical distribution to a non-canonical one.

Other results derived in section 4 are with regard to the transient values of the moment $\langle x_l^2 \rangle$ for the case of variable ω. A different point from the constant frequency case is that the relaxation time of the displacement moments before reaching its equilibrium value can be controlled by the statistical property of frequency noise ϵ and Δt_ω. According to exact equations with respect to the moments (61) and (65), the transient behaviour of the moments can be discussed by the eigenvalues of matrices L and $\mathcal{L}$. Furthermore, it is remarkable how very slow relaxation can be observed near the critical condition of divergence of the moment as shown in figure 3, which cannot be observed in the approximate formulation. The dynamics may be more essential than the equilibrium property when the slow dynamics is realized or the equilibrium values of the moments do not exist. So it is significant that the dynamical property of this problem can be derived from the matrices we have found.

Interesting results concerning the statistical property of the solution of the linear stochastic equation with two random coefficients have been obtained and some effects of the frequency fluctuation have been clarified without an approximation. However, the time correlation of the driving force and the frequency fluctuation between different time intervals were not considered. Further research on these will provide both an introduction to practical phenomena and further understanding of the relation between the characteristics of the noise and the stochastic behaviour of the system. The approximate formulation in which interaction between two decreasing modes is ignored can possibly be the starting point for considering the case of more complicated noise and thus be of use for further studies.

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